NATIONAL BUREAU OF STANDARDS REPORT

1665

A PROPERTY OF STRONGLY CONTINUOUS PROCESSES

by

Eugene Lukacs

U. S. DEPARTMENT OF COMMERCE
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1. <u>Introduction.</u>

The purpose of this report is to give a condition which assures that the increments of a stochastic process $y(t)$ are normally distributed. This condition is of interest in connection with the study of the fundamental random process. The fundamental random process - also called the Wiener process - affords a particularly simple model for certain phenomena. One of the properties of the fundamental random process is that its increments are normally and independently distributed (for a definition see [4] or [5]). In the present report it is shown that the normality of the increments follows from a certain continuity property.

2. <u>Definitions and statement of the theorem.</u>

In this section we give the definitions necessary to formulate the theorem.

The <u>increment</u> of a stochastic process $y(t)$ over the time interval $(t, t + \tau)$ is the random variable $y(t + \tau) - y(t)$.

A stochastic process is said to be a <u>process with independent increments</u> if the increments over non-overlapping time intervals are completely independent of each other.

A process $y(t)$ is said to be strongly continuous [5] in the interval $[a, b]$ if to every $\epsilon > 0$ and $\eta > 0$ there exists a $\delta = \delta(\epsilon, \eta)$ such that for every finite set S of points contained in $[a, b]$

(1) $P[\mathcal{E}(\delta, \epsilon, S)] \geqq 1 - \eta$.

Here $\mathcal{E}(\delta, \epsilon, S)$ is the event that the inequalities $|y(t_i) - y(t_k)| \leqq \epsilon$ are simultaneously satisfied for all pairs (t_i, t_k) with $|t_i - t_k| < \delta$ belonging to a finite set S of points contained in $[a, b]$.

The symbol $P[\ldots]$ stands here and in the following for the probability of the event $[\ldots]$ in the brackets.

We are now in a position to formulate the condition mentioned in the introductory section.

Theorem: Let $y(t)$ be a stochastic process and assume that

(i) $y(t)$ is a process with independent increments

(ii) $y(t)$ is strongly continuous in the interval $[a, b]$.

Then $y(b) - y(a)$ is normally distributed.

A theorem of this type is due to P. Levy [3] (theorem 16, 3). However it might be of interest to give a different proof using the notation and terminology of H. B. Mann's monograph [5].

3. Khintchine's theorem.

For the proof we need a theorem of A. Khintchine which gives conditions for the convergence to the normal distribution. This theorem is given in a book by Khintchine [2], published in the Russian language. The following formulation was taken from a paper by B. V. Gnedenko [1] which is available in an English translation.

We consider a sequence of sequences $X_{n1}, X_{n2}, \ldots X_{nk_n}$ $(n = 1, 2 \ldots$ ad inf$)$ of random variables which are independent within each sequence. The random variables $X_{n\nu}$ are said to be infinitesimal if for any $\epsilon > 0$ the relation $\lim_{n\to\infty} P[|X_{n\nu}| > \epsilon] = 0$ uniformly in ν $(1 \leqq \nu \leqq k_n)$.

We denote by $F_{n\chi}(x)$ the distribution function of the random variable $X_{n\chi}$ and state Khimtchine's theorem: If the distributions of the sums

(2) $T_n = X_{n1} + X_{n2} + \dots + X_{nk_n}$

of independent (within each sequence) infinitesimal random variables $X_{n\chi}(1 \leq \chi \leq k_n)$ converge to a limiting distribution, then the necessary and sufficient condition for the limiting distribution to be normal is that for any $\epsilon > 0$

$$(3) \lim_{n\to\infty} \sum_{\chi=1}^{k_n} \int_{|x|\geq\epsilon} dF_{n\chi}(x) = 0 .$$

For the proof the reader is referred to Gnedenko's paper [1].

4. Proof of the theorem stated in section 2

We consider a sequence $\{S_n\}$ of subdivisions of the interval $[a,b]$. For the sake of simplicity we let S_n be the subdivision $S_n = (t_0^n, t_1^n, t_2^n, \dots, t_n^n)$ of $[a,b]$ into n equal parts, that is we put $t_j^n = a + (b-a)\, j/n$ $(j = 0,1,2, \dots, n)$. We write $x_{n,j} = y(t_j^n) - y(t_{j-1}^n)$ for $j = 1,2, \dots, n$ and show first that the random variables $x_{n,j}$ are infinitesimal.

By assumption (ii) the process $y(t)$ is strongly continuous in $[a,b]$. It is therefore possible to determine for every $\epsilon > 0$ and $\eta > 0$ a $\delta = \delta(\epsilon, \eta)$ such that for every subdivision S_n

(4) $P[\mathcal{E}(\delta, \epsilon, S_n)] \geq 1 - \eta$

We denote by $P[|x_{n,j}| \leq \epsilon ; j = 1,2, \dots n]$ the probability that the n inequalities $|x_{n,j}| \leq \epsilon$ $(j = 1, 2, \dots n)$ hold simultaneously (*)

(*) If R_i $(i = 1,2, \dots n)$ are n events then $P[R_i ; i = 1,2, \dots n]$ means the probability that all n events occur simultaneously. Thus $P[R_i ; i = 1,2, \dots n] \geq 1-\eta$ means that the probability of the simultaneous occurrence of all n events is at least equal to $1-\eta$. This should be carefully distinguished from the statement $P[R_i] \geq 1-\eta$ $(i = 1,2, \dots n)$ which means that the probability of the occurrence of each single event is at least equal to ~~1/n~~; this statement does not imply anything about the probability of the joint occurrence of the n events.

$x(t_i)$

We next choose a number $N = N(\varepsilon, \eta)$ such that $N > \frac{(b-a)}{\delta(\varepsilon, \eta)}$.

For any $n \geq N$ the event $\mathcal{C}(\delta, \varepsilon, S_n)$ implies that the n inequalities $|x_{n,j}| \leq \varepsilon \ (j = 1, 2, \ldots n)$ are simultaneously satisfied. We conclude then from (4) that

(5) $1 - \eta \leq P[|x_{n,j}| \leq \varepsilon;\ j = 1, 2, \ldots n]$ and also

(6) $1 - \eta \leq P[|x_{n,j}| \leq \varepsilon]$ for $j = 1, 2, \ldots n$, if only $n \geq N$.

For every $\varepsilon > 0$ and $\eta > 0$ it is therefore possible to find an $N = N(\varepsilon, \eta)$ such that for $n \geq N$

(7) $P[|x_{n,j}| > \varepsilon] \leq \eta$ for $j = 1, 2, \ldots n$.

This shows that $P[|x_{n,j}| > \varepsilon]$ converges (uniformly in j) to zero as $n \to \infty$, or in other words the $x_{n,j}$ are infinitesimal random variables.

We consider the sequence of random variables

(8) $T_n = x_{n,1} + x_{n,2} + \ldots + x_{n,n}$

We have shown that T_n is the sum of infinitesimal random variables which are by assumption independent (within each sequence). Since

$$T_n = \sum_{j=1}^{n} x_{n,j} = \sum_{j=1}^{n} [y(t_j^n) - y(t_{j-1}^n)] = y(t_n^n) - y(t_0^n) = y(b) - y(a)$$

We see that the limiting distribution of the T_n is the distribution of the random variable $y(b) - y(a)$.

We have already shown that it is possible to find for every $\varepsilon > 0$ and $\eta > 0$ a $N = N(\varepsilon, \eta)$ such that $P[|x_{n,j}| \leq \varepsilon;\ j = 1, \ldots, n] \geq 1 - \eta$ for $n \geq N$.

Since the increments $x_{n,j}$ are independently distributed [by assumption (i)] it follows that also

(9) $P[|x_{n,j}| \leq \varepsilon;\ j = 1, 2, \ldots k] \geq 1 - \eta$

where ε and η are arbitrary positive numbers and $n \geq N$ while k is an integer less than n.

For any subdivision S_n of $[ab]$ into n equal parts we introduce the random variable

$$(10) \quad M_{abS_n} = \operatorname{Max}\,[\,|x_{n,1}|,\ |x_{n,2}|,\ \ldots\ldots,\ |x_{n,n}|\,].$$ — w.p.

The statement that the n inequalities $|x_{n,j}| \leqq \varepsilon$ for $j = 1,2,\ldots n$ hold simultaneously is equivalent to the statement $M_{abS_n} \leqq \varepsilon$.

Therefore

$$P[M_{abS_n} \leqq \varepsilon\,] = P[\,|x_{n,j}| \leqq \varepsilon\ ;\ j = 1,2,\ldots n] \geqq 1-\eta$$

and also

$$(11) \quad P[M_{abS_n} > \varepsilon\,] \leqq \eta\,,$$

for all $\varepsilon > 0$ and $\eta > 0$, provided that $n \geqq N$.

We introduce the following n events:

$B_1^{(n)}$ is the event that the inequality $|x_{n,1}| > \varepsilon$ holds.

$B_j^{(n)}$ (for $j = 2,3,\ \ldots n$) is the event that the j inequalities

$$|x_{n,1}| \leqq \varepsilon,\ |x_{n,2}| \leqq \varepsilon\ ,\ \ldots\ldots |x_{n,j-1}| \leqq \varepsilon\,,\ |x_{n,j}| > \varepsilon$$ hold

simultaneously. The events $B_1^{(n)}, B_2^{(n)}, \ldots\ldots B_n^{(n)}$ are mutually exclusive and exhaust all the cases for which $M_{abS_n} > \varepsilon$. /(a,-S)

Therefore we see from (11)

$$(12) \quad \eta \geqq P[M_{abS_n} > \varepsilon\,] = \sum_{j=1}^{n} P[B_j^{(n)}\,]$$ >

for all $\varepsilon > 0$ and $\eta > 0$, provided that $n \geqq N$.

Since the random variables $x_{n,1},\ldots\ldots,x_{n,n}$ are completely independent we have for $j = 2,\ldots n$

$$P[B_j^n] = P[\,|x_{n,k}| \leqq \varepsilon\ ;\ k = 1,2,\ldots(j-1)]\ P[\,|x_{n,j}| > \varepsilon\,].$$

It follows therefore from (9) that

$$(13) \quad P[B_j^n] \geqq (1-\eta)\ P[\,|x_{n,j}| > \varepsilon\,] \qquad (j = 2,\ldots,n)$$

for any $\varepsilon > 0$ and $\eta > 0$ if only $n \geqq N$ so that

$$(14) \quad \sum_{j=1}^{n} P[B_j^n\ ] \geqq (1-\eta) \sum_{j=1}^{n} P[\,|x_{n,j}| > \varepsilon\ ]$$

for any $\varepsilon > 0$ and $\eta > 0$ provided that $n \geqq N$.

From (12) and (14) we see that

$$(15) \qquad \eta/(1-\eta) \geq \sum_{j=1}^{n} P[|x_{n,j}| > \varepsilon]$$

for any $\varepsilon > 0, \eta > 0$ if $n \geq N$. Hence we conclude that

$$\lim_{n \to \infty} \sum_{j=1}^{n} P[|x_{n,j}| > \varepsilon] = 0 \quad \text{or}$$

$$(16) \qquad \lim_{n \to \infty} \sum_{j=1}^{n} \int_{|x|>\varepsilon} dF_{n,j}(x) = 0$$

But (16) is exactly Khinchine's condition (3) so that we have shown that the limiting distribution of the T_n, that is the distribution of $y(b) - y(a)$ is normal.

5. References.

1 . B. V. Gnedenko: Limit Theorems for Sums of Independent Random Variables. American Mathematical Society Translation No. 45, New York (1951). Translated from Uspehi Matem. Nauk 10, 115-165 (1944).

2 . A. Khintchine, Limiting distributions for sums of independent random variables. (in Russian) Moscow - Leningrad (1938).

3 . Paul Levy Processes stochastiques et mouvement Brownien. Gauthiers Villars, Paris 1948

4 . H. B. Mann Estimation of parameters in certain stochastic processes. Sankhya vol. 11, 97-106, (1951)

5 . H. B. Mann Introduction to the theory of stochastic processes depending on a continuous parameter.
N.B.S. report #1293 (1951) to be published as AMS 24.

THE NATIONAL BUREAU OF STANDARDS

Functions and Activities

The functions of the National Bureau of Standards are set forth in the Act of Congress, March 3, 1901, as amended by Congress in Public Law 619, 1950. These include the development and maintenance of the national standards of measurement and the provision of means and methods for making measurements consistent with these standards; the determination of physical constants and properties of materials; the development of methods and instruments for testing materials, devices, and structures; advisory services to Government Agencies on scientific and technical problems; invention and development of devices to serve special needs of the Government; and the development of standard practices, codes, and specifications. The work includes basic and applied research, development, engineering, instrumentation, testing, evaluation, calibration services and various consultation and information services. A major portion of the Bureau's work is performed for other Government Agencies, particularly the Department of Defense and the Atomic Energy Commission. The scope of activities is suggested by the listing of divisions and sections on the inside of the front cover.

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